Transonic rotational flow over a convex corner

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A singularity is encountered in the flow field about two-dimensional and axisymmetric bodies characterized by a sharp corner, where the fluid velocity becomes sonic. Investigation shows that the problem in question belongs, as do many other discontinuity problems, to the family of asymptotic or 'boundary-layer' phenomena of mathematical physics. The solution of a first approximation to the flow equations is given by a series in powers of a variable measuring the distance from the corner, with coefficients depending on an appropriate similarity variable. The leading coefficient of the series is independent of three-dimensional and rotationality effects, in complete analogy to the well-known solution of the corner problem in supersonic flow. Detailed results are presented for the leading singularity and for the first two corrections due to rotationality and axial symmetry of the flow.

1. Introduction

The uses of blunt wings and bodies for hypersonic flight has presented the aerodynamicist again with problems of the transonic type, wherein regions of subsonic and supersonic flow are encountered simultaneously. Among the many possible configurations, there is particular practical interest in those characterized by a sharp corner, which determines the location of the sonic point. This point of view is prompted by the experimental results of Ferri (1958) who showed that, by suitable location of the shoulder, one may reduce the heat transfer at the stagnation point on a blunt profile having a prescribed radius of curvature of the nose. From a theoretical point of view it is of interest to obtain information which, when combined with available methods such as those of Vaglio-Laurin & Ferri (1958), and Van Dyke (1958), permits the analysis of the flow field about general blunt-body shapes.

The investigation of the transonic portion of two-dimensional and axisymmetric rotational flows with a sonic singularity cannot take advantage of the hodograph transformation which has been the main tool of the aerodynamicist interested in classical two-dimensional irrotational transonic flow. For the present problem, however, analysis in the physical plane does not present excessive complications; in fact it has the advantage that one gains an immediate physical appreciation of the phenomenon which would be lost in the hodograph plane. The subsonic and supersonic portions of the flow are joined by a 'boundary layer' with a behaviour typical of the classical discontinuity phenomena common to many branches of physics (see, for example, von Kármán (1940), Carrier (1953), and Friedrichs (1955)). The leading term of the solution describing the flow in the 'boundary layer' predicts the same pressure distribution on the wall and the same sonic line shape as the leading singularity of Guderley's (1948) hodograph solution for the transonic corner problem in two-dimensional irrotational flow; the boundary-layer feature, and its bearing on the numerical analysis of complicated flow fields, could hardly be detected in the hodograph plane.

Transonic rotational flow around a sharp corner on an axisymmetric body has been investigated by Ho & Holt (1956), who used series expansions in powers of the distance from the shoulder with coefficients depending on the angular coordinate based on the shoulder. They found that the lowest-order terms correspond to a Prandtl–Meyer expansion; and they gave formal solutions for the first-order perturbations in the subsonic and in the supersonic regions of the flow. However, they did not investigate in detail the problem of matching the two regions. It is shown in §2 that direct matching cannot be carried out. This impossibility of matching together with the classical symptoms of (*a*) different analytic laws obtained to zero order in the subsonic and in the supersonic region, and (*b*) an infinite value of one velocity component at the boundary between the two regions, strongly suggest the occurrence of a boundary layer. Just to quote a few examples, the symptoms are the same as those encountered in the study of classical viscous boundary-layer theory, in the study of transonic flows by small-perturbation theory, and in the asymptotic theory of the wave equation at a caustic surface.

In general, boundary-layer analyses connected with solutions of non-linear partial differential equations are employed heuristically. In the present case one obtains *a posteriori* proof of validity from the following findings:

(*a*) As should be expected on physical grounds (see Vaglio-Laurin & Ferri (1958)), the leading term of the solution coincides with that pertaining to plane potential flow.

(*b*) The behaviour on the supersonic side is exactly that predicted by perturbation of a Prandtl–Meyer flow; however, the infinity in the perturbation velocities is eliminated.

Further heuristic substantiation of the approach may be found in the fact that the prevailing equations are the classical ones of transonic flow.

The details of the subject analysis are presented in the following sequence. The boundary-layer approach, the pertaining equations, their solution by means of a series of functions of an appropriate similarity variable, and the requirements thereon are first discussed (§2). A detailed study of the non-linear ordinary differential equation governing the leading singularity and the solution of this equation are the subject of §3. The linear equations governing the subsequent coefficients of the series solution and detailed results for the leading axisymmetric and rotational effects are then given (§4). Finally, the application of the results to the numerical analysis of the supersonic flow about blunt nosed bodies with a sonic shoulder is discussed in §5.

2. The governing equations

The difficulty associated with a straightforward analysis of the corner singularity is apparent when one particularizes the results of Ho & Holt (1956) to plane potential flow. With the notation of these authors shown in figure 1, one obtains for the first-order coefficients in the subsonic region

$$U_1(\theta) = A_s[1+\cos 2(\theta+\alpha)], \quad (1a)$$

$$V_1(\theta) = -A_s \sin 2(\theta+\alpha), \quad (1b)$$

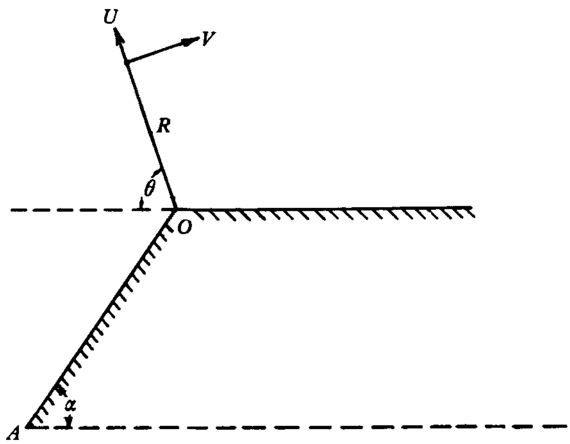

FIGURE 1. Schematic diagram and notation for preliminary considerations.

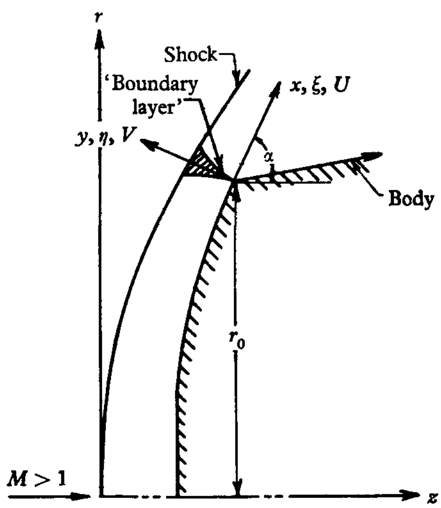

FIGURE 2. Schematic diagram of typical sonic shoulder and reference co-ordinate system.

and in the supersonic region

$$U_1(\theta) = \text{const.}\ [\cos(\lambda\phi)]^{(2\lambda^2+1)/2\lambda^2}[\sin(\lambda\phi)]^{\frac{1}{2}}, \quad (2a)$$

$$V_1(\theta) = \tfrac{1}{2}U_1'(\theta), \quad (2b)$$

with $$\lambda^2 = \frac{\gamma-1}{\gamma+1}, \quad \phi = \theta+\alpha-\tfrac{1}{2}\pi.$$

Matching at the sonic line requires

$$(V_1)_{\phi=0} = 0,$$

or, in view of $(2a, b)$ zero perturbation in the flow field. The different behaviour with ϕ of the subsonic and of the supersonic perturbations suggests the same conclusion; yet the perturbations in question are certainly valid outside of the sonic region. The situation is then typical of an asymptotic phenomenon; a boundary-layer approach is suggested.

As in many classical problems, the parameter of the asymptotic expansion does not occur in the differential equations governing the flow considered here. This parameter must then be related to one of the characteristic quantities in the problem, say the departure of the local velocity from the sonic value, by the requirement that the approximate boundary-layer equations have all the qualitative features necessary to describe the phenomenon on hand, namely, a transonic flow. With the notation of figure 2 (x-co-ordinate across the boundary layer), the asymptotic expansion is easily carried out along the following lines:

(*a*) New independent variables are introduced

$$\bar{\xi} = x/\delta, \quad \bar{\eta} = y, \tag{3}$$

where δ is a quantity which may represent the thickness of the boundary layer, and which tends to zero as the magnitude of the velocity perturbation from sonic at the edges of the layer tends to zero.

(*b*) The velocity field is described by small perturbations superposed on a parallel sonic stream in the x-direction. However, the orders of magnitude of the perturbations in the $\bar{\xi}$ and $\bar{\eta}$ directions can be different; we denote them by ϵ_i and ν_i, respectively, to get

$$\frac{U}{a_*} = 1+\epsilon_1 U_1(\bar{\xi},\bar{\eta})+\epsilon_2 U_2(\bar{\xi},\bar{\eta})+\ldots, \tag{4a}$$

$$\frac{V}{a_*} = \nu_1 V_1(\bar{\xi},\bar{\eta})+\nu_2 V_2(\bar{\xi},\bar{\eta})+\ldots. \tag{4b}$$

(*c*) In view of the asymptotic Prandtl–Meyer expansion behaviour to be matched on the supersonic side, we set $\nu_1^2 = \epsilon_1^3$.

(*d*) In the axisymmetric case we assume $\delta \ll r$, that is, the thickness δ of the boundary layer at any point is much smaller than the radial distance of the point from the axis. Under these conditions, which are met by all present practical configurations, the effect of transverse curvature is negligible, and one can set within the boundary layer

$$r = r(\eta) = r_0+\eta\cos\alpha. \tag{5}$$

The analogy with the classical problem of the viscous boundary layer on a body of revolution is quite evident.

On this basis one can proceed to the expansion of the equations of motion for homoenergetic flow written in the form

$$a^2 \operatorname{div}\mathbf{q} = \mathbf{q}\,.\nabla(\tfrac{1}{2}q^2), \tag{6a}$$

$$a^2 = \frac{\gamma+1}{2}a_*^2 - \frac{\gamma-1}{2}q^2, \tag{6b}$$

$$\boldsymbol{\omega}\times\mathbf{q} = \frac{\rho^{\gamma-1}}{\gamma-1}\nabla(p_\infty\rho_\infty^{-\gamma}\pi^{1-\gamma}), \tag{6c}$$

$$\pi = \exp\left(\frac{S_\infty - S}{R}\right), \tag{6d}$$

where $\mathbf{q}$ denotes the local velocity vector, a the local speed of sound, $\boldsymbol{\omega}$ the vorticity vector, p the pressure, ρ the density, S the entropy, and R the gas constant. The subscript ∞ refers to free-stream conditions.

It is of interest to notice that, since we are concerned with a local boundary-layer phenomenon, the ideal gas analysis to be developed here can be applied also to problems involving flows of real gases in thermodynamic equilibrium. When (6*a*, *c*) are expanded along the lines stipulated in (*a*) to (*d*) above, one obtains the following leading equations

$$\epsilon_1^{\frac{3}{2}}\left(V_{1\bar{\eta}}+j\frac{\cos\alpha}{r_0+\bar{\eta}\cos\alpha}V_1\right)-\epsilon_1^2\delta^{-1}(\gamma+1)\,U_1\,U_{1\bar{\xi}}=0, \tag{7a}$$

$$\epsilon_1 U_{1\bar{\eta}}-\epsilon_1^{\frac{3}{2}}\delta^{-1}V_{1\bar{\xi}}=\mu_1 D\left(\frac{2}{\gamma+1}\right)^{1/(\gamma-1)}\frac{\rho_{0\infty}}{\rho_r}\frac{a_*}{\gamma}\frac{d\pi}{d\psi}, \tag{7b}$$

where the quantities j and D are defined by

$$\left.\begin{array}{lll} j=0, & D=1 & \text{for two-dimensional flow,}\\ j=1, & D=\left(1+\dfrac{\bar{\eta}\cos\alpha}{r_0}\right)r_0 & \text{for axisymmetric flow,}\end{array}\right\} \tag{8a}$$

the stream function ψ is defined by

$$\frac{\partial\psi}{\partial x}=-\frac{\rho}{\rho_r}VD,\quad \frac{\partial\psi}{\partial y}=\frac{\rho}{\rho_r}UD, \tag{8b}$$

and the factor μ_1 denotes the order of magnitude of the local entropy level. Physically significant results are obtained from (7*a*, *b*) only for

$$\delta=\epsilon_1^{\frac{1}{2}},\quad \mu_1=\epsilon_1. \tag{9}$$

Thus, we find that the thickness of the boundary layer is proportional to the square root of the perturbation velocity in the $\bar{\xi}$-direction; upon completion of the solution we shall give a quantitative definition of the thickness δ. The second relation in (9), namely, the identity of the orders of magnitude of velocity perturbations and entropy gradients, could have been stipulated *a priori*.

Equations (7*a*, *b*) can be simplified further. First, we notice that for $\alpha=\frac{1}{2}\pi$ the axisymmetric problem degenerates into the strictly two-dimensional problem; with this understanding we introduce new independent variables ξ, η defined by

$$\left.\begin{array}{lll} \xi=\bar{\xi}, & \eta=\bar{\eta} & \text{for two-dimensional flow,}\\ \xi=\dfrac{\cos\alpha}{r_0}\bar{\xi}, & \eta=\dfrac{\cos\alpha}{r_0}\bar{\eta} & \text{for axisymmetric flow.}\end{array}\right\} \tag{10}$$

Secondly, we observe that, within the boundary-layer approximation, the entropy gradients can be expressed by

$$\frac{d\pi}{d\psi}=\sum_{i=0}^{\infty}c_i\eta^i. \tag{11}$$

Then $(7a, b)$ can be rearranged in the final form

$$V_{1\eta} + j\left[1 + \sum_{i=1}^{\infty} (-1)^i \eta^i\right] V_1 - (\gamma+1) U_1 U_{1\xi} = 0, \tag{12a}$$

$$U_{1\eta} - V_{1\xi} = \sum_{i=0}^{\infty} K_i \eta^i, \tag{12b}$$

where

$$\left.\begin{aligned} K_i &= \left(\frac{2}{\gamma+1}\right)^{1/(\gamma-1)} \frac{\rho_{0\infty}}{\rho_r} \frac{a_*}{\gamma} c_i && \text{for two-dimensional flow,} \\ K_i &= \left(\frac{2}{\gamma+1}\right)^{1/(\gamma-1)} \frac{r_0^2}{\cos\alpha} \frac{\rho_{0\infty}}{\rho_r} \frac{a_*}{\gamma} (c_{i-1} + c_i) && \text{for axisymmetric flow } (i \geqslant 1). \end{aligned}\right\} \tag{13}$$

The final form of the equations is, from a mathematical point of view, the same as that obtained in the analysis of transonic flow. The essential non-linearity that permits the description of mixed flows is retained.

The solution of $(12a, b)$ by means of series expansions in powers of the distance from the shoulder, with coefficients depending on an appropriate similarity variable, will now be sought. To this effect we let

$$\left.\begin{aligned} U_1 &= (\gamma+1)^{-\frac{1}{2}} \sum_{m=0}^{\infty} \eta^{d_m} u_m(\zeta), \\ V_1 &= \sum_{m=0}^{\infty} \eta^{e_m} v_m(\zeta), \\ \zeta &= (\gamma+1)^{-\frac{1}{2}} \xi \eta^{-p}. \end{aligned}\right\} \tag{14}$$

Substitution of (14) into (12) yields

$$d_0 = 2p-2, \quad d_1 = 1, \quad d_2 = 2p-1, \quad d_3 = 2,$$

$$e_0 = 3p-3, \quad e_1 = p, \quad e_2 = 3p-2, \quad e_3 = 1+p, \text{etc.}$$

One can easily prove that the exponent p in the equation for the similarity variable ζ must satisfy the requirement $1 \leqslant p < 1{\cdot}5$. The changes of flow properties along the body surface in the supersonic region must satisfy the compatibility condition for a characteristic line, namely (cf. Ferri 1954)

$$\cot\mu \frac{dq}{q} - d\theta + \sin\mu\cos\mu \frac{dS}{\gamma R} - \frac{\sin\mu \sin\theta}{\cos(\theta+\mu)} \frac{dz}{r} = 0. \tag{15}$$

Since the boundary layer extends into the supersonic region, the solution must comply with (15); effects of either three-dimensionality or rotationality of the flow must vanish at $\eta = 0$. Hence, one requires $d_1 > d_0$ and $p < 1{\cdot}5$. Also, if one combines the equation expressing the slope of the sonic line

$$\left(\frac{d\eta}{d\xi}\right)_{\text{sonic}} = \frac{1}{q}\left(\frac{\partial q}{\partial \theta}\right)_{S=\text{const.}}$$

with (15), one obtains $(d\eta/d\xi)_{\text{sonic}} = \infty$ and, therefore, $p \geqslant 1$. The equations to be satisfied by the leading terms $(n = m = 0)$ in the series (14) follow immediately

$$(3p-3)v_0 - p\zeta v_0' - u_0 u_0' = 0, \tag{16a}$$

$$(2p-2)u_0 - p\zeta u_0' - v_0' = 0. \tag{16b}$$

Upon solution of ($16a, b$) (see § 3 for details), one finds $p = 1{\cdot}25$. The difference between subsequent powers of η in the series (14) is then equal to $\tfrac{1}{2}$; the linear equations to be satisfied by the pertaining coefficients can be written in the following general form

$$(d_m+p-1)\,v_m-p\zeta v_m'-(u_0u_m'+u_0'u_m)=\sum_{i=1}^{m-1}u_iu_{m-i}'-j\sum_{i=1}^{\frac{1}{2}m}(-1)^{-1}v_{(m-2i)},\qquad(17a)$$

$$d_mu_m-p\zeta u_m'-v_m'=K_{\frac{1}{2}(m-1)}(\gamma+1)^{\frac{1}{3}}.\qquad(17b)$$

As $\zeta\to+\infty$ (supersonic side), the solutions of the systems (16) and (17) are required to behave like a Prandtl–Meyer expansion and perturbations thereof. A second set of boundary conditions consistent with the given body profile is imposed on the subsonic side. The detailed solutions of the systems (16) and (17) are reviewed in the following two sections. Several relevant considerations are similar to those made by Guderley & Yoshihara (1949) in their study of axisymmetric transonic flows.

3. The leading singularity

The flow described by ($16a, b$) is irrotational; and so a potential can be introduced

$$\Phi=\eta^{3p-2}g(\zeta),\qquad(18)$$

such that

$$u_0=g',\quad v_0=(3p-2)\,g-p\zeta g'.\qquad(19)$$

Equation ($16b$) is then identically satisfied, while ($16a$) becomes

$$(g'-p^2\zeta^2)\,g''+5p(p-1)\,\zeta g'-(3p-3)\,(3p-2)\,g=0.\qquad(20)$$

The non-linear ordinary differential equation (20) has a group property; namely, if $g = G(\zeta)$ is a solution then other solutions are given by $g = k^{-3}G(k\zeta)$, where k is an arbitrary constant. When an equation has such a property, its order can be lowered by one through an appropriate change of variables. In the present case one introduces new variables s and t

$$s=\zeta^{-3}g,\quad t=\zeta^{-2}g',\qquad(21)$$

to obtain the first-order equation

$$\frac{dt}{ds}=\frac{(3p-2)\,(3p-3)\,s-p(3p-5)\,t-2t^2}{(t-3s)\,(t-p^2)},\qquad(22)$$

and the inverse transformation law

$$\log\zeta=\int\frac{ds}{t-3s}.\qquad(23)$$

Actually the change of variables is advantageous only in as far as (22) lends itself more readily to a qualitative study of the integral curves; on this basis one can proceed directly to the numerical integration of (20).

Investigation of (22) must provide the following information: (a) location of the singular points; (b) physical significance of these points; (c) structure of the integral curves in their neighbourhood; (d) general trend of the integral curves. The singular points in the finite portion of the (s, t)-plane are

$$s=0,\ \tfrac{1}{3},\ \tfrac{5}{3}p^3(3p-2)^{-1},\quad t=0,\ 1,\ p^2.\qquad(24)$$

The behaviour at infinity in the (s, t)-plane is studied by the conical transformation

$$s = x_1/x_0, \quad t = x_2/x_0 \tag{25}$$

and by subsequent intersection of the conical surfaces representing the integral curves with a selected plane, say $x_1 = 1$. With this choice of the plane of intersection, the integral curves of (22) are projected into those of the equation

$$\frac{dx_2}{dx_0} = \frac{x_2(x_2-3)(x_2-p^2x_0)+2x_2^2+p(3p-5)x_2x_0-(3p-2)(3p-3)x_0}{x_0(x_2-3)(x_2-p^2x_0)}, \tag{26}$$

with singular points at

$$x_0 = 0,\ 0, \quad x_1 = 1,\ 1, \quad x_2 = 0,\ 1. \tag{27}$$

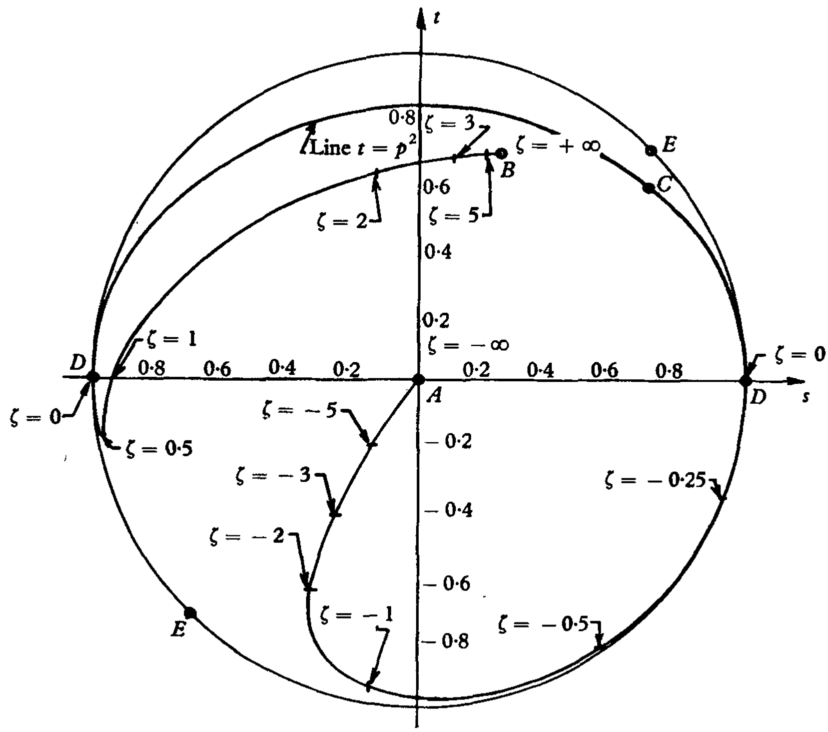

FIGURE 3. Spherical projection of the (s, t)-plane and of the curve representing the coefficient of the leading singularity. (The indicated scale of ζ is that for sonic velocity at $\zeta = 1$.)

If one selected $x_2 = 1$ as projection plane, no other singularity would appear. Hence, we conclude that all the singular points of (22) are those listed in (24) and (27). In the following we shall examine them in detail. The schematic diagram of the (s, t)-plane shown in figure 3 will be of some assistance in these considerations.

Point $s = 0, t = 0$

In the neighbourhood of this point the differential equation can be simplified to the form

$$\frac{dt}{ds} = \frac{(3p-2)(3p-3)s-p(3p-5)t}{p^2(3s-t)}. \tag{28}$$

The integral curves are then given by

$$(t-\alpha s)^{\mu}(t-\beta s)^{\nu} = C_1, \tag{29}$$

where

$$\alpha = \frac{1}{p}(3p-2), \quad \beta = \frac{1}{p}(3p-3),$$

$$\mu = -2, \quad \nu = 3.$$

The point in question is a node. In the neighbourhood of the same,

$$\zeta = C_2\left[t-\frac{1}{p}(3p-2)s\right]^{-\frac{1}{3}p}, \tag{30a}$$

$$\Phi_\xi = (\gamma+1)^{(\frac{2}{3}-p)/p}\,\xi^{2(p-1)/p}\,tC_2^{\frac{3}{2}}\left[t-\frac{1}{p}(3p-3)s\right]^{-\frac{2}{3}}, \tag{30b}$$

$$\Phi_\eta = -(\gamma+1)^{(1-p)/p}\frac{C_2^{3/p}}{p}\,\xi^{3(p-1)/p}. \tag{30c}$$

Since $\zeta \to \infty$, the point represents one of the edges of the boundary layer. The velocity components $(30b, c)$ do not match the desired behaviour on the supersonic side; hence, the edge in question must be the subsonic one ($\zeta \to -\infty$). For a regular body shape on the subsonic side the velocity component Φ_η must have a non-singular asymptotic behaviour as $\zeta \to -\infty$; therefore, the solution $(30c)$ is not acceptable in general. This condition leads to selecting among the solutions (29) that characterized by $C_1 = \infty$ and by

$$\left.\begin{aligned} t = \frac{1}{p}(3p-2)s, \quad \zeta = C_3 s^{-\frac{1}{2}p}, \\ \Phi_\xi = (\gamma+1)^{(\frac{2}{3}-p)/p}\frac{3p-2}{p}C_3^{2/p}\,|\xi|^{2(p-1)/p}, \quad \Phi_\eta = 0. \end{aligned}\right\} \tag{31}$$

Correspondingly, one finds for the function g the asymptotic expansion

$$g = (-\zeta)^{3-2/p}\sum_{i=0}^{\infty} A_i(-\zeta)^{-2i/p}, \tag{32}$$

with A_0 arbitrary ($A_0 = -C_3^{2/p}$) and

$$A_1 = \left(3-\frac{2}{p}\right)^2\left(\frac{1}{p}-1\right)A_0^2, \tag{33a}$$

$$A_2 = \frac{1}{6p^3}(2-3p)(3-2p)(4-3p)A_0A_1, \tag{33b}$$

$$A_3 = \frac{1}{15p^3}[6(2-p)^2(2-3p)A_0A_2+(2-p)(4-3p)^2A_1^2], \tag{33c}$$

$$A_4 = \frac{1}{28p^3}[(2-3p)(5-2p)(8-3p)A_0A_3+3(2-p)(4-3p)(5-2p)A_1A_2], \tag{33d}$$

$$A_5 = \frac{1}{45p^3}[2(2-3p)(3-p)(10-3p)A_0A_4 + 2(3-p)(4-3p)(8-3p)A_1A_3+9(2-p)^2(3-p)A_2^2]. \tag{33e}$$

The sign of A_0 (> 0) is determined by the condition that $\Phi_\xi < 0$ (subsonic flow) as $\zeta \to -\infty$. Correspondingly, one moves along the integral curve in the (s, t)-plane in the direction of negative t. The magnitude of A_0 (i.e. the scale of ζ) remains arbitrary in accordance with the group property mentioned above; we shall select it so as to obtain sonic velocity ($g' = 0$) at $\zeta = 1$.

Point $s = \frac{1}{3}$, $t = 1$

In the neighbourhood of this point equation (22) is simplified by introducing variables

$$\sigma = s - \tfrac{1}{3}, \quad \tau = t - 1$$

to get

$$\frac{d\tau}{d\sigma} = \frac{(3p-2)(3p-3)\sigma - (3p^2 - 5p + 4)\tau}{(1-p^2)(\tau - 3\sigma)}. \tag{34}$$

The integral curves are then given by

$$(\tau - \alpha\sigma)^\mu (\tau - \beta\sigma)^\nu = C_1, \tag{35}$$

where

$$\alpha = (3p-2)(p-1)^{-1}, \quad \beta = 3(p-1)(p+1)^{-1},$$
$$\mu = (p+1)(7p-5)^{-1}, \quad \nu = 6(p-1)(7p-5)^{-1}.$$

For the range of interest $1 \leqslant p < 1{\cdot}5$, it is $1 \geqslant \mu > 0$ and $0 \leqslant \nu < 1$; hence, the point in question is a saddle of the integral surface (35). Along the principal directions $\tau = k\sigma$ ($k = \alpha, \beta$) through the saddle-point (these are the only two integral curves through the point), one obtains

$$\zeta = C_2 \sigma^{1/(k-3)}, \tag{36a}$$

$$\Phi_\xi = (\gamma+1)^{-\frac{1}{3}} \eta^{2p-2} \zeta^2 = (\gamma+1)^{-1} \frac{\xi^2}{\eta^2}, \tag{36b}$$

$$\Phi_\eta = -\tfrac{2}{3}\eta^{3p-3}\zeta^3 = -\tfrac{2}{3}(\gamma+1)^{-1}\frac{\xi^3}{\eta^3}. \tag{36c}$$

If $k = \alpha$, ζ tends to zero and the point $s = \frac{1}{3}$, $t = 1$ represents the η-axis. The sonic line ($\Phi_\xi = 0$) coincides with the η-axis; however, the flow is supersonic on either side of this line. Hence, the solution represented by the α integral curve must be rejected. If $k = \beta$, ζ tends to infinity, and the point in question represents either edge of the boundary layer, depending on the sign of the constant C_2. The appropriate choice is $\zeta \to +\infty$ since the velocity components (36*b*, *c*) exhibit the desired asymptotic behaviour of a Pradtl–Meyer expansion. Thus, we find that the solution for the leading singularity must be represented by the integral curve entering the singular point in the β-direction. Correspondingly, we obtain for the function g the asymptotic expansion

$$g = \zeta^3 \left[\frac{1}{3} + \sum_{i=0}^{\infty} B_i \zeta^{(\beta-3)(i+1)}\right], \tag{37}$$

with B_0 arbitrary and

$$B_1 = \beta^2(1-\beta) B_0^2 \{(3-2\beta)[2(1-\beta)(1-p^2) - p(7p-5)] - (3p-2)(3p-3)\}^{-1}, \tag{38a}$$

$$B_2 = -\beta(3-2\beta)(5-3\beta) B_0 B_1 \times \{(6-3\beta)[(5-3\beta)(1-p^2) - p(7p-5)] - (3p-2)(3p-3)\}^{-1}, \tag{38b}$$

$$B_3 = -2(2-\beta)\,[6\beta(2-\beta)\,B_0 B_2 - (3-2\beta)\,B_1^2] \times \{(9-4\beta)\,[4(2-\beta)\,(1-p^2) - p(7p-5)] - (3p-2)\,(3p-3)\}^{-1}, \quad (38c)$$

$$B_4 = -(11-5\beta)\,[\beta(9-4\beta)\,B_0 B_3 - 3(3-2\beta)\,(2-\beta)\,B_1 B_2] \times \{(12-5\beta)\,[(11-5\beta)\,(1-p^2) - p(7p-5)] - (3p-2)\,(3p-3)\}^{-1}, \quad (38d)$$

$$B_5 = -(7-3\beta)\,[2\beta(12-5\beta)\,B_0 B_4 - 2(3-2\beta)\,(9-4\beta)\,B_1 B_3 - 9(2-\beta)^2 B_2^2] \times \{3(5-2\beta)\,[2(7-3\beta)\,(1-p^2) - p(7p-5)] - (3p-2)\,(3p-3)\}^{-1}. \quad (38e)$$

The sign of B_0 (<0) is determined by the condition that the velocity component Φ_ξ be decreasing as one moves inside the boundary layer from the supersonic edge; in the (s,t)-plane one is then limited to move along the β-direction in the sense of decreasing t. Again the magnitude of B_0 (scale of ζ) can be fixed arbitrarily in view of the group property of (20).

Point $s = \frac{5}{3}p^3[3p-2]^{-1},\ t = p^2$

In the neighbourhood of this point the differential equation (22) is simplified by introducing variables

$$\sigma = s - \tfrac{5}{3}p^3(3p-2)^{-1}, \quad \tau = t - p^2$$

to get

$$\frac{d\tau}{d\sigma} = \frac{p(7p-5)\,(3p-2)\,\tau - (3p-2)^2\,(3p-3)\,\sigma}{2p^2(p+1)\,\tau}. \quad (39)$$

As for the previous singular point considered, the integral curves are formally given by (35) with

$$\alpha = \frac{3p-2}{2p}, \quad \beta = 3(3p-2)\frac{p-1}{p(p+1)},$$

$$\mu = 6(1-p)\,(7-5p), \quad \nu = (1+p)\,(7-5p)^{-1}.$$

For the range of interest ($1 \leqslant p < 1{\cdot}5$), μ and ν are of opposite sign; hence, the point in question is a node.

Several comments are warranted at this stage. First, we observe that any integral curve intersecting the line $t = p^2$ at $s \neq \frac{5}{3}p^3(3p-2)^{-1}$ represents a solution with a limiting line. Indeed, at the point of intersection one finds an extremum of ζ, since

$$\frac{ds}{dt} = 0 \quad \text{and, therefore,} \quad d(\log\zeta) = 0,$$

Any such solution must be rejected. Secondly, we note that any point on the line $t = p^2$ represents a first family (positive slope) characteristic line of the system (16a, b) in the physical plane since, for $t = p^2$, the two quantities

$$\left(\frac{d\eta}{d\xi}\right)_{\zeta=\text{const.}} = (\gamma+1)^{-\frac{1}{3}}\,(p\eta^{p-1}\zeta)^{-1},$$

$$\left(\frac{d\eta}{d\xi}\right)_{\text{charact.}} = (\gamma+1)^{-\frac{1}{3}}\,\Phi_\xi^{-\frac{1}{2}}$$

become identical; however, consistent with the aforementioned occurrence of limiting lines, the compatibility conditions along the characteristic, namely

$$d(\Phi_\eta) - \tfrac{2}{3}(\gamma+1)^{\frac{1}{3}}\,d(\Phi_\xi^{\frac{3}{2}}) = 0,$$

is identically satisfied only when the representative point is at $s = \frac{5}{3}p^3(3p-2)^{-1}$. Solutions represented by integral curves passing through the latter point must also be rejected on the following grounds: When $p > 1$ the quantities α and β are positive while the quantity $(t-3s)$ is negative; then

$$\frac{d}{dt}(\log \zeta) = (t-3s)^{-1}\frac{ds}{dt}$$

is smaller than zero. An observer proceeding in the sense of increasing ζ across the boundary layer described by such a solution, would encounter a compression in the neighbourhood of the characteristic that maps at the singular point; this situation is impossible in the subject expansion flow.

Thus, we conclude that the integral curve of interest must lie all below the line $t = p^2$.

Point $x_0 = 0,\ x_2 = 0$

In the neighbourhood of this point equation (26) can be simplified to the form

$$\frac{dx_2}{dx_0} = \frac{p(6p-5)\,x_0x_2 - x_2^2 - (3p-2)\,(3p-3)\,x_0}{3x_0(p^2x_0 - x_2)}, \tag{40}$$

which is not reducible to the standard type encountered in the previous cases. We introduce then an additional assumption; namely, $x_2 \gg x_0$, and reduce (40) to

$$\frac{dx_2}{dx_0} = \frac{x_2}{3x_0} + (p-1)\,(3p-2)\frac{1}{x_2},$$

which can immediately be integrated to obtain

$$x_2 = x_0^{\frac{1}{3}}[C_1 + 6(3p-2)\,(p-1)\,x_0^{\frac{1}{3}}]^{\frac{1}{2}} \tag{41}$$

consistent with the assumption. If we assumed $x_0 \gg x_2$, we could also integrate the equation readily; however, the result would not be consistent with the simplifying hypothesis. We conclude that (41) represents the integral curves of (40) within the neighbourhood in question. A plot of these integral curves is shown in figure 4. From (23), (25) and (41) one obtains

$$\zeta = C_2 s^{-\frac{1}{4}}. \tag{42}$$

Thus, as one proceeds along an integral curve through the point in question, the variable ζ goes through zero and changes of sign. It may then be conjectured that such will be the case for the integral curve of interest.

Point $x_0 = 0,\ x_2 = 1$

In the neighbourhood of this point (26) is simplified by introducing the variable $\sigma = (x_2 - 1)$ to obtain

$$\frac{d\sigma}{dx_0} = -\frac{\sigma}{2x_0} + (2p^2 - 5p + 3)$$

with the integral curves

$$x_2 - 1 = C_1 x_0^{-\frac{1}{2}} + \tfrac{2}{3}(2p^2 - 5p + 3)\,x_0. \tag{43}$$

From (23), (25) and (43) one obtains

$$\zeta = C_2 \exp(s^{-\frac{1}{2}}).$$

For s and t tending to either $+\infty$ or $-\infty$, ζ remains finite; hence, g and g' must tend to infinity, given infinite velocity. The integral curves passing in the vicinity (or through) this point must be rejected.

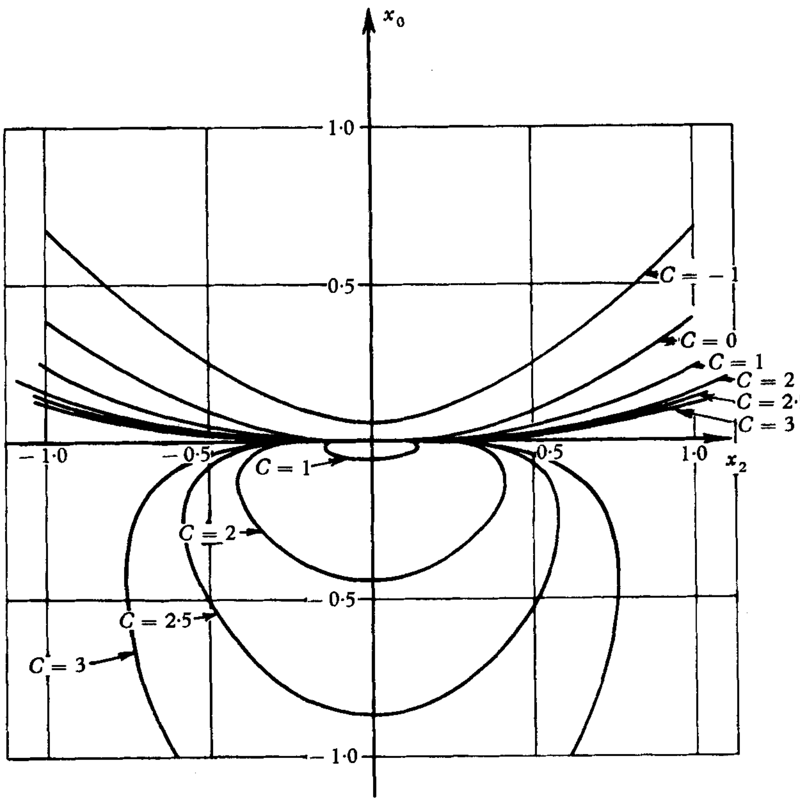

FIGURE 4. Integral curves in the neighbourhood of the singular point $x_0 = 0$, $x_2 = 0$.

From the above investigation of the singular points of (22), we conclude that the desired integral curve should have the trend shown in figure 3. The value of the exponent p, which defines the similarity variable ζ, is determined by the condition that the integral curves through the points $s = 0, t = 0$ and $s = \frac{1}{3}, t = 1$ are described by the same asymptotic law (41) as $s \to \infty$ and $t \to \infty$ ($\zeta \to 0$).

The numerical work is expedited if one operates directly on (20) along the following lines: (*a*) Two numerical integrations, both proceeding toward $\zeta = 0$, are carried out for several values of p; one integration proceeds from large negative ζ with initial values given by the series (32), and one from large positive ζ with initial values given by the series (37). (*b*) The quantities s and t are computed in the neighbourhood of $\zeta = 0$ (on both sides) from the definitions (21), and are used to determine the constant C_1 in (41). (*c*) The correct p is that for which the two values of C_1, as determined for $\zeta > 0$ and for $\zeta < 0$, are identical. In this way one finds

$$p = 1{\cdot}25, \quad C_1 = 1{\cdot}4369.$$

The scale of ζ can then be chosen and matched on both sides of $\zeta = 0$. We have selected it so that the sonic line ($g' = 0$) is described by the equation

$$\zeta = (\gamma+1)^{-\frac{1}{3}}\xi\eta^{-\frac{5}{4}} = 1.$$

With this choice of scale the coefficients A_0, B_0 and C_2 in (32), (37) and (42) become, respectively,

$$A_0 = 1{\cdot}9444, \quad B_0 = -3{\cdot}2636, \quad C_2 = -1{\cdot}2489.$$

Corresponding values of the functions g, g', g'', and $v_0 = [(3p-2)g - p\zeta g']$ are plotted in figure 5 for the range $-3 \leqslant \zeta \leqslant 3$. Outside of this range all the aforementioned functions are accurately described by the series (32) and (37).

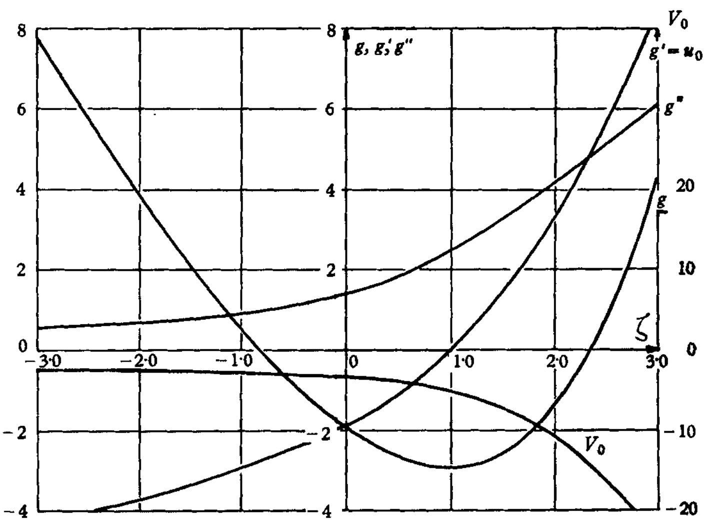

FIGURE 5. The function $g(\zeta)$ and its derivatives.

Several aspects of the solution just obtained warrant some comment. We emphasize again that the coefficient of the leading term in the series (14), which describes the flow in the boundary layer, is independent of axial symmetry and rotationality, and that it exhibits the behaviour of a Prandtl–Meyer expansion on the supersonic side. From the asymptotic behaviour of Φ_η for $\zeta \to -\infty$ [equation (31)], we find that this leading term is consistent with a flat wall on the subsonic side; we shall see in the following section that the effect of wall shape is manifested in subsequent terms of the series.

We can then compare some results of the present analysis with those obtained by Guderley's (1948) hodograph solution for two-dimensional irrotational flow over a corner with a flat wall on the subsonic side. From (31) we obtain on the subsonic side

$$\Delta p \sim \Phi_\xi \sim |\xi|^{0{\cdot}4}.$$

From the definitions of the variable ζ and of the velocity component $\Phi_\eta \sim \theta$ ($\theta \equiv$ flow direction measured from the ξ-axis), we obtain on the sonic line ($\zeta = 1$, $g' = 0$)

$$\theta \sim \eta^{\frac{3}{4}} \sim \xi^{\frac{3}{5}}.$$

Both results coincide with those given by Guderley. It can also be shown that the present solution degenerates into the well-known Prandtl–Meyer expansion when the subsonic flow is uniform (see Vaglio-Laurin 1959).

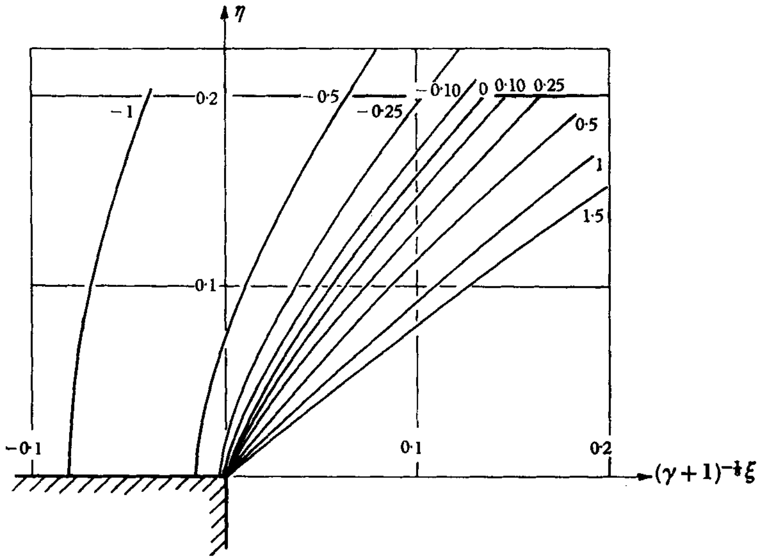

FIGURE 6. Flow described by leading singularity. Lines of constant velocity $(\gamma+1)^{\frac{1}{2}}\,\Phi_\xi = \text{const.}$

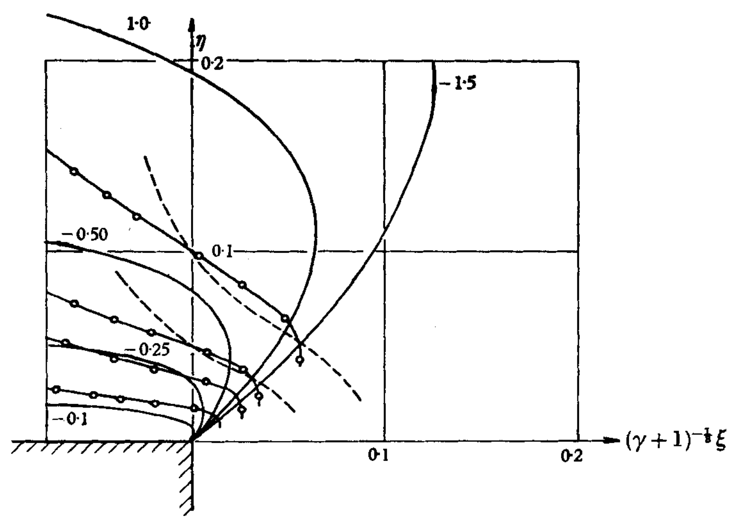

FIGURE 7. Flow described by basic singularity: ——— Lines of constant flow deflexion $\theta = \Phi_\eta = \text{const.}$; —○— approximate streamlines $(\gamma+1)^{\frac{1}{2}}\,(d\eta/d\xi) = \Phi_\eta$; ---- actual streamlines $(\gamma+1)^{\frac{1}{2}}\,(d\eta/d\xi) = \Phi_\eta(1+\Phi_\xi)^{-1}$.

Some features of the flow field described by the leading singularity are shown in figures 6 and 7, where isobars, isoclines, and streamlines are plotted for the case with sonic line at $\zeta = 1$.

4. The coefficients of higher-order terms

Subsequent terms in the series (14) must be determined when the analysis of an actual flow field (either plane or axisymmetric), with given velocity and entropy distributions at infinity, is required. Upon determination of the coefficients of the higher-order terms, the behaviour of the subsonic flow in the neighbourhood of the sonic velocity can be prescribed; these boundary conditions, together with conditions at infinity and on the surface of the body, are sufficient to determine the subsonic flow field and, in particular, the magnitude of the aforementioned coefficients at the subsonic edge of the boundary layer ($\zeta \to -\infty$). Corresponding velocity distributions at the supersonic edge of the layer are provided readily by the subject solutions; thus, initial values for classical perturbations superposed on a Prandtl–Meyer flow are obtained, while the matching difficulty attached to a straightforward perturbation procedure is overcome.

The coefficients of subsequent terms in the series (14) are given by the solution of systems of linear equations written in general form at (17). Since the equations are linear, the pertaining solutions can be divided into a complementary function and a particular integral, which we shall denote by $u_m^{(0)}$, $v_m^{(0)}$ and $u_m^{(p)}$, $v_m^{(p)}$, respectively. Here we shall discuss the determination of the complementary functions, and give the particular integrals for the coefficients representing the leading effects of rotationality and axial symmetry, namely, the coefficients of η^{d_1} and η^{d_2} ($d_1 = 1$, $d_2 = 1{\cdot}5$).

In connexion with the analysis of the complementary solutions we can introduce potential functions

$$\phi_m = \eta^{d_m+p} f_m(\zeta), \tag{44}$$

such that

$$u_m^{(0)} = f_m', \quad v_m^{(0)} = (d_m+p)f_m - p\zeta f_m'. \tag{45}$$

The homogeneous equation associated with (17*b*) is then identically satisfied, while the homogeneous equation associated with (17*a*) becomes

$$(g'-p^2\zeta^2)f_m'' + [g'' + (2d_m+p-1)p\zeta]f_m' - (d_m+p)(d_m+p-1)f_m = 0. \tag{46}$$

Since the quantity $(g'-p^2\zeta^2)$ is never equal to zero (negative for all values of ζ), (46) has only two singular points at $\zeta = \pm\infty$. By standard methods two independent solutions for $\zeta \to -\infty$ are found:

$$f_{m1} = (-\zeta)^{(d_m/p)+1} \sum_{i=0}^{\infty} a_{mi}^{(1)} (-\zeta)^{-2i/p}, \tag{47a}$$

$$f_{m2} = (-\zeta)^{\{(d_m-1)/p\}+1} \sum_{i=0}^{\infty} a_{mi}^{(2)} (-\zeta)^{-2i/p}, \tag{47b}$$

with $a_{m0}^{(1)}$, $a_{m0}^{(2)}$ arbitrary and, for $i \geqslant 1$,

$$a_{mi}^{(1)} = \frac{d_m+2p-2i}{p^3 i(2i-1)} \sum_{j=0}^{i-1} (d_m+p-2j)(i-j-\tfrac{3}{2}p)A_{i-j-1}a_{mj}^{(1)}, \tag{48a}$$

$$a_{mi}^{(2)} = \frac{d_m+2p-(2i+1)}{p^3 i(2i+1)} \sum_{j=0}^{i-1} (d_m+p-1-2j)(i-j-\tfrac{3}{2}p)A_{i-j-1}a_{mj}^{(2)}. \tag{48b}$$

For $\zeta \to +\infty$, two independent solutions are given by

$$\tilde{f}_{m1} = \zeta^{(d_m+p)/(p-1)} \sum_{i=0}^{\infty} b_{mi}^{(2)} \zeta^{-6i/(p+1)}, \tag{49a}$$

$$\tilde{f}_{m2} = \zeta^{(d_m+p-1)/(p+1)} \sum_{i=0}^{\infty} b_{mi}^{(2)} \zeta^{-6i/(p+1)}, \tag{49b}$$

with $b_{m0}^{(1)}$, $b_{m0}^{(2)}$ arbitrary and, for $i \geqslant 1$,

$$b_{mi}^{(1)} = -\frac{1}{2i}\left[\frac{d_m+p}{p-1} - \frac{6i}{p+1} + 1\right][(6i-1)(1-p) + 2(d_m+p)]^{-1}$$
$$\times \sum_{j=0}^{i-1} [p-1-2(i-j-1)]\left[\frac{d_m+p}{p-1} - \frac{6j}{p+1}\right] B_{i-j-1} b_{mj}^{(1)}, \tag{50a}$$

$$b_{mi}^{(2)} = -\frac{1}{2i(p+1)^2}[d_m+2p-6i][(6i-1)(1-p) - 2(d_m+p)]^{-1}$$
$$\times \sum_{j=0}^{i-1} [p-1-2(i-j-1)][d_m+p-1-6j] B_{i-j-1} b_{mj}^{(2)}. \tag{50b}$$

The solutions $(47a, b)$ give the following asymptotic behaviour of the velocity components at the subsonic edge of the boundary layer ($\zeta \to -\infty$):

$$(\gamma+1)^{-\frac{1}{3}} \eta^{d_m} u_{m1}^{(0)} = -(\gamma+1)^{-(d_m+p)/3p}\left(\frac{d_m}{p}+1\right)|\xi|^{d_m/p}, \tag{51a}$$

$$\eta^{e_m} v_{m1}^{(0)} = 0, \tag{51b}$$

$$(\gamma+1)^{-\frac{1}{3}} \eta^{d_m} u_{m2}^{(0)} = -(\gamma+1)^{-(d_m+p-1)/3p}\left(\frac{d_m-1}{p}+1\right)\eta\,|\xi|^{(d_m-1)/p}, \tag{52a}$$

$$\eta^{e_m} v_{m2}^{(0)} = (\gamma+1)^{-(d_m+p-1)/3p}\,|\xi|^{(d_m+p-1)/p}. \tag{52b}$$

The solutions $(49a, b)$ give the following asymptotic behaviours at the supersonic edge ($\zeta \to +\infty$):

$$(\gamma+1)^{-\frac{1}{3}} \eta^{d_m} u_{m1}^{(0)} = (\gamma+1)^{-(d_m+p)/3(p-1)} \frac{d_m+p}{p-1} \eta^{-1}\left(\frac{\xi}{\eta}\right)^{(d_m+1)/(p-1)}, \tag{53a}$$

$$\eta^{e_m} v_{m1}^{(0)} = (\gamma+1)^{-(d_m+p)/3(p-1)} \eta^{-1}\left(\frac{\xi}{\eta}\right)^{(d_m+p)/(p-1)}, \tag{53b}$$

$$(\gamma+1)^{-\frac{1}{3}} \eta^{d_m} u_{m2}^{(0)} = (\gamma+1)^{-(d_m+p-1)/3(p+1)} \frac{d_m+p-1}{p+1} \eta^{2(d_m+p-1)/(p+1)}\left(\frac{\xi}{\eta}\right)^{(d_m-2)/(p+1)}, \tag{54a}$$

$$\eta^{e_m} v_{m2}^{(0)} = (\gamma+1)^{-(d_m+p-1)/3(p+1)} \frac{d_m+2p}{p+1} \eta^{2(d_m+p-1)/(p+1)}\left(\frac{\xi}{\eta}\right)^{(d_m+p-1)/(p+1)}. \tag{54b}$$

One can immediately verify that, as $\zeta \to +\infty$, the solution $u_{m2}^{(0)}$, $v_{m2}^{(0)}$, described by $(54a, b)$, exhibits the behaviour appropriate to perturbations superposed on a Prandtl–Meyer flow; hence, such a solution merges smoothly with that proposed by Ho & Holt (1956) at the supersonic edge of the boundary layer. Vice versa, it can be seen from $(53a, b)$ that the second solution [cf. $(49a)$] for $\zeta \to +\infty$ must be rejected; indeed, infinite velocities at the corner would be obtained, in contrast with the physical boundary condition that the flow exactly on the body exhibit the behaviour of a Prandtl–Meyer expansion, as described by the leading singularity.

Thus, one boundary condition is obtained for the second-order equation (46) and, therefore, for the general system (17). The second condition is imposed at the subsonic edge of the boundary layer by requiring that the asymptotic behaviour of the η-component of velocity $[\eta^{e_m}(v_{m2}^{(0)} + v_m^{(p)})]$ be such as to satisfy tangency at the surface of the given body. The solution is then completely determined.

The complementary functions pertaining to (46) must be evaluated numerically. Initial conditions at large positive values of ζ are obtained (except for one arbitrary multiplying constant) from the series (49*b*): the solution is then continued by numerical integration toward large negative values of ζ, where it is matched with a linear combination of the series (47*a*, *b*). Upon evaluation of the particular integral, the arbitrary constant is finally determined by imposing the tangency condition on the subsonic side.

Inspection of the asymptotic behaviour of the velocity components described by the complementary functions [see (51*a*, *b*), (52*a*, *b*), and (54*a*, *b*)] leads to the

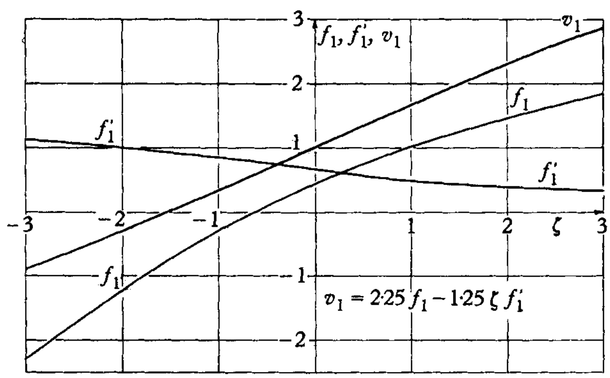

FIGURE 8. The functions $f_1(\zeta)$, $f_1'(\zeta)$ and $v_1(\zeta)$ associated with the effect of rotationality.

interesting conclusion that the subsonic and the supersonic flow are described by series involving different powers of η. Under these conditions neither straightforward expansion procedures nor gross numerical schemes can be used to analyse the flow around the sonic shoulder on a body.

Having in mind the application of the present results to the analysis of the hypersonic flow field about blunt-nosed bodies characterized by a sonic shoulder, we have determined in detail the solutions pertaining to the second and third terms in the series (14); the leading effects of entropy gradient at the wall and of axial symmetry of the flow are found thereby. It is felt that such information may be sufficient to describe the flow field within a reasonably large neighbourhood of the corner; outside of this neighbourhood the calculations could be continued by a suitable numerical scheme without requiring an extremely refined network. In this connexion, one must not overlook the fact that the present results have been obtained on the basis of a transonic small-perturbation theory and are applicable to regions of the flow field within which this approximation is valid.

The complementary functions for $d_1 = 1$ and $d_2 = 1{\cdot}5$ (satisfying the boundary conditions on the supersonic side) are plotted in figures 8 and 9 for the range

$-3 \leqslant \zeta \leqslant 3$. The solution f_1 contributes to the terms representing the effect of a uniform vorticity; the solution f_2 gives the analogous contribution for the effect of axial symmetry. Outside of the range $-3 \leqslant \zeta \leqslant 3$ the functions are accurately predicted by the series ($47a, b$) and ($49b$). The particular integrals to be used in

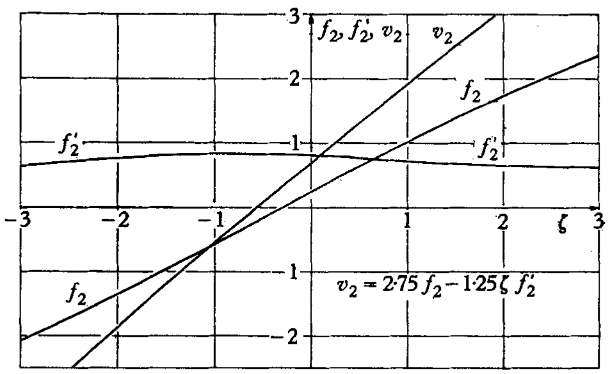

FIGURE 9. The functions $f_2(\zeta)$, $f_2'(\zeta)$ and $v_2(\zeta)$ associated with the effect of axial symmetry.

connexion with these complementary functions are easily found by inspection of the pertaining systems of equations. From the general form (17) one obtains

$$\left.\begin{aligned} pv_1 - p\zeta v_1' - (u_0 u_1' + u_0' u_1) &= 0, \\ (p+1)\, u_1 - p\zeta u_1' - v_1' &= (\gamma+1)^{\frac{1}{3}} K_0, \end{aligned}\right\} \tag{55}$$

$$\left.\begin{aligned} (p+\tfrac{1}{2})\, v_2 - p\zeta v_2' - (u_0 u_2' + u_0' u_2) &= u_1 u_1' - jv_0, \\ \tfrac{3}{2}u_2 - p\zeta u_2' - v_2' &= 0, \end{aligned}\right\} \tag{56}$$

for $d_1 = 1$ and $d_2 = 1{\cdot}5$, respectively. A particular integral of the system (55) is given by

$$u_1^{(p)} = 0, \quad v_1^{(p)} = -(\gamma+1)^{\frac{1}{3}} K_0 \zeta, \tag{57}$$

with the asymptotic behaviour

$$\zeta \to \pm\infty, \quad \eta^p v_1^{(p)} = \mp K_0 |\xi|.$$

A particular integral of the system (56), accounting only for the axisymmetric contribution (term jv_0), is given by

$$u_2^{(p)} = \tfrac{1}{10}[p\zeta g'' - 2(p+1)\, g'], \tag{58a}$$

$$v_2^{(p)} = -\tfrac{1}{10}[p^2\zeta^2 g'' - (5p+1)\, p\zeta g' + (3p-1)(3p+2)\, g], \tag{58b}$$

with the following asymptotic behaviour. For $\zeta \to -\infty$,

$$(\gamma+1)^{-\frac{1}{3}} \eta^{1{\cdot}5} u_2^{(p)} = \tfrac{14}{25}(\gamma+1)^{-\frac{7}{15}} A_0 \eta |\xi|^{\frac{3}{5}}, \tag{59a}$$

$$\eta^{1{\cdot}75} v_2^{(p)} = -\tfrac{2}{5}(\gamma+1)^{-\frac{7}{15}} A_0 |\xi|^{\frac{7}{5}}, \tag{59b}$$

while for $\zeta \to +\infty$,

$$(\gamma+1)^{-\frac{1}{3}} \eta^{1{\cdot}5} u_2^{(p)} = -\tfrac{1}{5}(\gamma+1)^{-1} \eta \left(\frac{\xi}{\eta}\right)^2, \tag{60a}$$

$$\eta^{1{\cdot}75} v_2^{(p)} = \tfrac{1}{15}(\gamma+1)^{-1} \eta \left(\frac{\xi}{\eta}\right)^3. \tag{60b}$$

It will be noticed that identical asymptotic expressions of the η-component of velocity at the subsonic edge of the boundary layer are obtained from the complementary functions and from the particular integrals in the two cases considered. Hence, the tangency condition at the wall in the subsonic region can be satisfied. It should also be noticed that at the supersonic edge of the boundary layer the particular integrals depend on lower powers of η as compared to the complementary functions; thus, the contribution of the former is predominant in the immediate neighbourhood of the corner.

Finally, it should be recognized that the higher-order terms under consideration predict a finite ξ-component of velocity on the η-axis (line $\zeta = 0$). The asymptotic solutions ($54a, b$) are valid for large ζ and $\xi/\eta > 0$; therefore, no difficulty is associated with the cases characterized by $d_m < 2$.

With the information obtained thus far one may, for example, set up a procedure for the numerical determination of hypersonic flow fields about blunt bodies with a sonic shoulder. This problem is discussed in the following section.

5. Outline for application

A representative configuration of a blunt body with a sonic shoulder is shown in figure 10. The problem of determining in detail the flow field about blunt bodies of general shape can only be attacked by numerical methods. Two approaches can be followed to this effect, namely: (*a*) the 'integral method' as applied, for example, by Belotserkovskii (1958); (*b*) the 'forward integration method' as proposed, for example, by Vaglio-Laurin & Ferri (1958).

For the particular category of bodies considered here, an extension of the integral method is required to obtain the appropriate behaviour at the sonic velocity. Several difficulties are encountered in connexion with the choice of an appropriate co-ordinate system for the region adjacent to the corner, and with the determination of the number of terms required for a reasonably accurate description of the flow at a general station.

No procedural difficulty arises when the forward integration method is used. Between the alternatives of proceeding either from a prescribed shock shape or from an estimated pressure distribution on the body, the latter is preferable on the grounds of numerical work involved and of available rules for carrying out initial estimates. This point of view is corroborated by a study of Kendall's (1959) experiments; observed pressure distributions upstream of the shoulder are in agreement with the present results and with the Mach number independence principle over a range extending to low supersonic flight conditions ($M_\infty \geqslant 2$), while correlations of observed shock shapes remain uncertain.

If the forward integration analysis of the subsonic region is to proceed from the body toward the shock, one prescribes a pressure distribution consistent with the singularity at the shoulder and then analyses simultaneously the elliptic and the transonic regions up to the limiting characteristic; the final solution is obtained by iteration. When the pressure distribution in the neighbourhood of the shoulder is given, the scale of the subject singularity can readily be determined in accord with the transonic similarity rule appropriate to the simplified governing equations of § 2;* thus, consistent initial conditions can be prescribed on a contour enclosing the shoulder point and extending into the supersonic region. The procedure and the transformation suggested by Vaglio-Laurin & Ferri (1958) appear to be useful for the numerical calculations. The transformation in question stretches the region adjacent to the shoulder and, therefore, requires a relatively less refined network in this neighbourhood; also, *a priori* knowledge of the

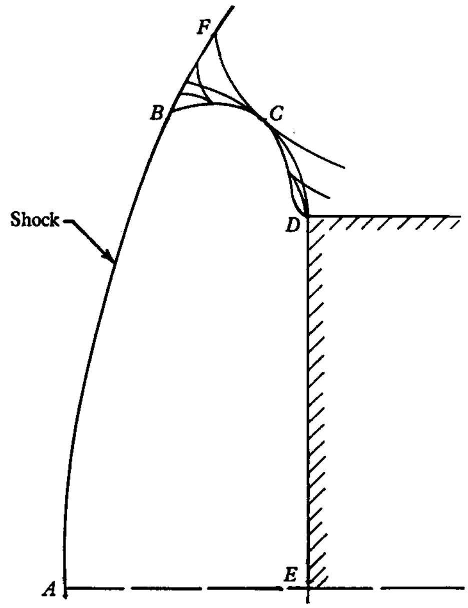

FIGURE 10. Schematic diagram of flow field about a blunt body with sonic shoulder. $ABCDEA$ = subsonic region; $DCFBCD$ = transonic region.

location of streamlines and shock in the transformed plane facilitates the estimate of a consistent entropy distribution and the application of boundary conditions at the shock front. Determination of complete flow fields using initial data based on the present solution, and subsequent comparison of theoretically determined shapes of sonic line and shock against experiments, will be of interest. Such an investigation will establish the domain of validity of the present results for practical configurations and flight conditions.

6. Concluding remarks

A critical review of the difficulties associated with a straightforward series-expansion approach to the present problem has led to the introduction of a boundary layer joining subsonic and supersonic regions. The pertaining equations have been given; a series solution therefor has been indicated. Upon determination of the leading terms in the series, the validity of the boundary layer concept has been established on the basis of: (*a*) physical criteria, such as the predicted behaviour on the solid boundary; (*b*) comparison with the known hodograph solution for two-dimensional irrotational flow; (*c*) consideration of the 'boundary-

* This similarity rule also manifested itself in the analysis of the leading singularity through arbitrariness of scale for the similarity variable ζ.

layer' flow at the supersonic edge, where the behaviour of a Prandtl–Meyer expansion and superposed perturbations is attained.

Detailed results have been presented for the leading singularity and for the first two corrections due to rotationality and axial symmetry of the flow. It has been shown that the boundary conditions at the supersonic edge essentially determine the behaviour of the coefficients in the series solution, while the conditions at infinity and the body profile in the subsonic region determine the thickness of the boundary layer, the nature of the series describing the flow in the layer, and the magnitude of the velocity components encountered therein.

As an example, the application of the present results in the analysis of the hypersonic flow about blunt bodies characterized by a sonic shoulder has been discussed. Bodies within this category are of practical interest, since, by suitable location of the shoulder, one may reduce the heat transfer at the stagnation point.

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Appendix

Numerical values of the leading coefficients for the series about the various singular points discussed in the analysis are compiled here. The reader interested in results outside of the asymptotic range can find detailed tabulations of the functions g, f_1 and f_2 in a report by Vaglio-Laurin (1959).

Leading singularity. When the scale of ζ is selected so that the sonic line coincides with $\zeta = 1$, the coefficients of the series (32), relative to the singularity $\zeta = -\infty$, are

$$A_0 = 1{\cdot}9444, \quad A_1 = -1{\cdot}4821, \quad A_2 = 0{\cdot}053795,$$
$$A_3 = -0{\cdot}017573, \quad A_4 = 0{\cdot}0095676, \quad A_5 = -0{\cdot}0067092.$$

The coefficients of the series (37), relative to the singularity $\zeta = +\infty$, are

$$B_0 = -3{\cdot}2636, \quad B_1 = -0{\cdot}056356, \quad B_2 = 0{\cdot}015895,$$
$$B_3 = -0{\cdot}0096063, \quad B_4 = 0{\cdot}0081281, \quad B_5 = -0{\cdot}0083370.$$

Higher-order terms. Numerical values are provided for the leading terms, respectively characterized by $d_1 = 1$ and $d_2 = 1{\cdot}5$. With the selected scale of ζ; the coefficients of the series $(47a, b)$, relative to the singularity at $\zeta = -\infty$, are

	$d_1 = 1$		$d_2 = \frac{3}{2}$	
i	$a_{1i}^{(1)}$	$a_{1i}^{(2)}$	$a_{2i}^{(1)}$	$a_{2i}^{(2)}$
0	1	1	1	1
1	−2·9394	−0·18144	−4·7901	−0·50804
2	−0·035555	0·035555	0	0·027654
3	−0·014193	−0·018493	−0·056770	−0·013763
4	0·018962	0·013276	0·046353	0·010353
5	−0·020263	−0·011346	−0·040683	−0·0093181

The coefficients of the series (49*b*), relative to the singularity $\zeta = +\infty$, are

	$d_1 = 1$	$d_2 = 1{\cdot}5$
i	$b_{1i}^{(2)}$	$b_{2i}^{(2)}$
0	1	1
1	0·040292	0·038902
2	−0·015134	−0·013883
3	0·011670	0·010644
4	−0·012053	−0·010998
5	0·014615	0·013356

The complementary functions f_m satisfying the boundary condition at $\zeta \to +\infty$ are linearly related to the solutions f_{m1}, f_{m2} about $\zeta \to -\infty$

$$d_1 = 1, \quad f_1 = -0{\cdot}11918 f_{11} - 0{\cdot}65264 f_{12};$$
$$d_2 = 1{\cdot}5, \quad f_2 = 0{\cdot}080053 f_{21} - 0{\cdot}52709 f_{22}.$$

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